$$\mathcal{O}(\mathcal{E}) = \widehat{\mathrm{Sym}}(\mathcal{E}^\vee)$$

$$\mathcal{E}^\vee \cong \overline{\mathcal{E}}_c^!$$

$$E^! := E^\vee \otimes \mathrm{Dens}(M)$$

There's a natural pairing

$$ev : \mathcal{E}_c \otimes \mathcal{E}^! \longrightarrow \mathbb{R}$$

$$e \quad \lambda \otimes |vol| \longmapsto \int_M \lambda(e)\, |vol| < \infty$$

$\mathcal{O}(\mathcal{E})$ is an algebra under mult.,

even though $\mathcal{E}^\vee$ is distributional.

$$\mathrm{Sym}^n(\mathcal{E}^\vee) := \Big(\underbrace{\mathcal{E}^\vee \hat{\otimes}_\pi \mathcal{E}^\vee \hat{\otimes}_\pi \mathcal{E}^\vee \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{E}^\vee}_{n \text{ times}}\Big)_{S_n}$$

S_n symmetric group coinvariants

$$= (\mathcal{E}^\vee)^{n\hat{\otimes}_\pi} / S_n$$

$$I \in \mathrm{Sym}^m(\mathcal{E}^\vee), \quad J \in \mathrm{Sym}^n(\mathcal{E}^\vee)$$

Pick a reps

$$\tilde{I} \in (\mathcal{E}^\vee)^{m\hat{\otimes}_\pi}, \quad \tilde{J} \in (\mathcal{E}^\vee)^{n\hat{\otimes}_\pi}$$

$$\tilde{I} \,\hat{\otimes}_\pi\, \tilde{J} \in (\mathcal{E}^\vee)^{(m+n)\hat{\otimes}_\pi} \longrightarrow \mathcal{E}^{(m+n)\hat{\otimes}_\pi}/S_n$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ni IJ$$

Use the iso w/ $\overline{\mathcal{E}}^!_c$.

To tensor spaces like this, we need $\hat{\otimes}_\beta$ 'completed bornological tensor product'.

Roughly: A 'bornological vector space' has a notion of bounded subsets, in contrast to TVS, which have open subsets.

$\hat{\otimes}_\pi$ ~ good for TVS called nuclear spaces.

$\hat{\otimes}_\beta$ ~ good for more general TVS, including distributional sections & compactly supp. secs.

Every locally convex Hausdorff TVS has an associated bornology.

When $\mathcal{E}$ & $\mathcal{F}$ are nuclear

$$\mathcal{E} \hat{\otimes}_\pi \mathcal{F} \cong \mathcal{E} \hat{\otimes}_\beta \mathcal{F}$$

$\mathcal{E} = \Gamma(M, E)$, $\mathcal{F} = \Gamma(N, F)$

$$\mathcal{E} \hat{\otimes}_\pi \mathcal{F} \cong \Gamma(M \times N, E \boxtimes F)$$

$$\cong \mathcal{E} \hat{\otimes}_\beta \mathcal{F}$$ because they're nuclear

However for $\hat{\otimes}_\beta$

$$\mathcal{E}_c \hat{\otimes}_\beta \mathcal{F}_c \cong \Gamma_{\text{proper}}(M \times N, E \boxtimes F)$$

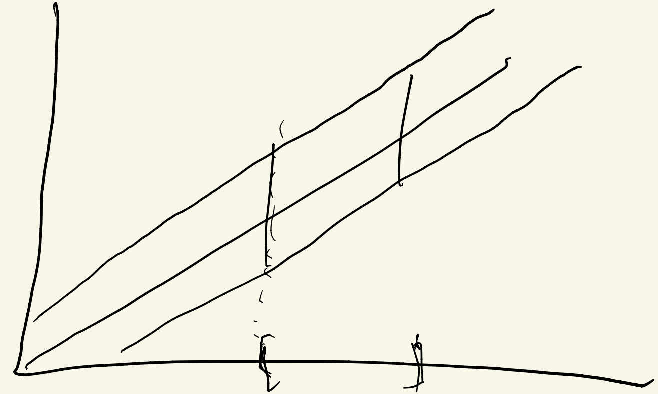
$X \subseteq M \times N$ is <u>proper</u> if

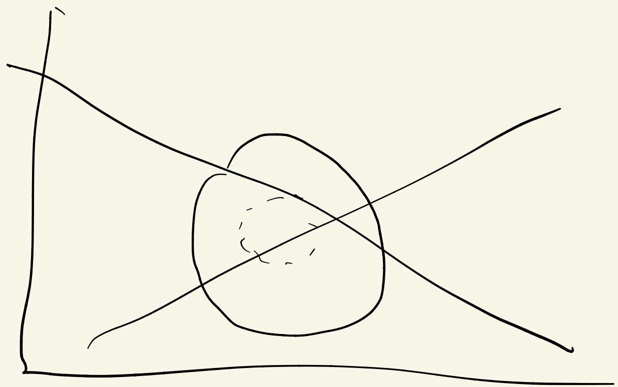
for $K \subseteq M$ compact,

$\left(\pi_1\big|_X\right)^{-1}(K)$ is compact

; similarly for $K \subseteq N$.

Ex Proper M





$\mathbb{R}$

$$\mathcal{E}^\vee \cong \overline{\mathcal{E}}^!_c$$

$$\mathcal{E}^{\vee\, n \hat{\otimes}_\pi} \cong \left(\overline{\mathcal{E}}^!_c\right)^{n \hat{\otimes}_\beta}$$

$$\cong \overline{\Gamma}_{prop}\left(M^n, \left(E^!\right)^{n \boxtimes}\right)$$

$$I \in \mathrm{Sym}^n\left(\mathcal{E}^\vee\right)$$

$$\tilde{I} \in \left(\mathcal{E}^\vee\right)^{n \hat{\otimes}_\pi}$$

$$\tilde{I} \in \Gamma_c\left(M^n, (E^!)^{\boxtimes n}\right)$$

$$\tilde{I}(x_1, \dots, x_n) \otimes \tilde{J}(x_{n+1}, \dots, x_{n+m})$$

Again, take quotient by S_{m+n}.

$e \in \mathcal{E}$, get 1st order diff op

$$\partial_e : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$$

Defined by contraction,

$$I \in \mathrm{Sym}^n(\mathcal{E}^\vee)$$

$$\tilde{I} \in \overline{\Gamma}_{prop}\left(M^n, (E^!)^{\boxtimes n}\right)$$

$$\partial_e \tilde{I} = \sum_{i=1}^{n} \varepsilon_i \int_{x_i \in M} \tilde{I}(x_1, \ldots, x_i, \ldots x_n)\, e(x_i)$$

Koszul sign

ε_i is the sign obtained from moving e through the components $a_1, \ldots, a_{i-1}$.

Pass to the quotient by S_n action.

Fix a basis for the typical fiber of E, get indices on my sections:

$$e(x) \rightsquigarrow e^a(x)\, T_a$$

$$\tilde{I} = \tilde{I}_{a_1 a_2 \cdots a_n}$$

pairs to density

$$\int_{x_i \in M} \tilde{I}_{a_1 a_2 \cdots a_i \cdots a_n} \; e^{a_i}$$

Alternatively, include a density index on $\tilde{I}$.

Similarly, given a section

$$P \in \mathcal{E} \hat{\otimes}_\pi \mathcal{E}$$

get a 2nd order differential operator

$$\partial_P : \mathcal{O}(\mathcal{E}) \longrightarrow \mathcal{O}(\mathcal{E})$$

again defined by contraction.

In particular:

$$\Delta_L := -\partial_{K_L}$$

scale L BV Laplacian.

Induces the scale L P.B.:

$$\{I, J\}_L = \Delta_L(IJ) - (\Delta_L I)J - (-1)^{|I|} I \Delta_L(J)$$

QME_L

$$QI + \frac{1}{2}\{I, I\}_L + \hbar \Delta_L I = 0$$

for $\quad I \in \mathcal{O}(\mathcal{E})[[\hbar]]$.

Propagator

$$P(\varepsilon, L) = \int_\varepsilon^L (Q^{GF} \otimes 1) K_\ell \, d\ell$$

RG flow

$$W(P(\varepsilon, L), I) = \text{Feynman diagram expansion}$$

$$= \sum_{\substack{\text{stable} \\ \text{conn} \\ \gamma}} \frac{1}{\mathrm{Aut}(\gamma)} w_\gamma(P(\varepsilon, L), I)$$

$P(\varepsilon, L) \in \mathcal{E} \hat{\otimes}_{\pi} \mathcal{E}$

$\rightsquigarrow \quad \partial P(\varepsilon, L)$

$$= \hbar \log \left(\exp\left(\hbar \partial_{P(\mathcal{E},L)}\right) \circ \exp\left(I/\hbar\right)\right)$$

Lemma $I \in \mathcal{O}^{+}(\mathcal{E})[[\hbar]]$ satisfies $QME_{\mathcal{E}}$ iff $W(P(\mathcal{E},L), I)$ satisfies the QME_L.

K_ℓ is the heat kernel for the generalized Laplacian

$$D = [Q, Q^{GF}]$$

$$P(\varepsilon, L) = \int_\varepsilon^L (Q^{GF} \otimes 1) K_\ell \, d\ell$$

$$P(\varepsilon, L) \in \mathcal{E} \hat{\otimes}_\pi \mathcal{E}$$

$$\rightsquigarrow \quad \partial_{P(\mathcal{E},L)} : \mathcal{O}(\mathcal{E}) \longrightarrow \mathcal{O}(\mathcal{E}),$$

defined by contraction.

$$e \in \mathcal{E} \quad \rightsquigarrow \quad \partial_e : \mathcal{O}(\mathcal{E}) \longrightarrow \mathcal{O}(\mathcal{E})$$

$$\partial_e = i_e = e \neg$$

$$\sum e \otimes f \in \mathcal{E} \hat{\otimes}_\pi \mathcal{E} \quad \rightsquigarrow \sum \partial_e \partial_f = \partial_{\sum e \otimes f}$$

Pf (of lemma)

Assume $I \in \mathcal{O}(\mathcal{E})[[\hbar]]$ satisfies the QME_ε.

Key observation:

$P(\varepsilon, L)$ is the kernel of the operator

$$\int_\varepsilon^L Q^{GF} e^{-\ell D} \, d\ell$$

e^{-lD} is the heat operator w/ kernel K_l.

$$P(\varepsilon, L) = \int_\varepsilon^L (Q^{GF} \otimes 1) K_l \, dl$$

is the kernel of

$$\int_\varepsilon^L Q^{GF} e^{-lD} \, dl$$

Compute

$$\left[Q, \int_{\varepsilon}^{L} Q^{GF} e^{-\ell D} \, d\ell \right]$$

$$= \int_{\varepsilon}^{L} \left[Q, Q^{GF} e^{-\ell D} \right] d\ell$$

$$= \int_{\varepsilon}^{L} [Q, Q^{GF}] \, e^{-\ell D} \, d\ell$$

$$= \int_{\varepsilon}^{L} D e^{-\ell D} \, d\ell$$

$$= e^{-\varepsilon D} - e^{-LD}$$

So: $\int_{\varepsilon}^{L} Q^{GF} e^{-\ell D} \, d\ell$ is a chain homotopy from $e^{-\varepsilon D}$ to e^{-LD}.

Use identity:

$$F : \mathcal{E}_c \longrightarrow \overline{\mathcal{E}}$$

$$K_F \in \overline{\mathcal{E}} \hat{\otimes}_\pi \overline{\mathcal{E}}$$

$$Q K_F = K_{[Q,F]}$$

Apply where $F = \int_\varepsilon^L Q^{GF} e^{-lD} dl$

$$[Q, F] = e^{-\varepsilon D} - e^{-LD}$$

K is linear:

$$QK_F = K_{[Q,F]} = K_\varepsilon - K_L$$

$$QK_F = QP(\varepsilon, L)$$

$$\boxed{Q P(\varepsilon, L) = K_\varepsilon - K_L}$$

We apply the ∂.

Somehow, Costello concludes

$$[\partial_{P(\varepsilon, L))} Q] = \partial_{K_\varepsilon} - \partial_{K_L}$$

Recall: $\Delta_\ell = -\partial_{K_\ell}$

$$[\partial_{P(\varepsilon,L)}, Q] = \partial_{K_\varepsilon} - \partial_{K_L}$$

$$= \Delta_L - \Delta_\varepsilon$$

$\partial_{P(\varepsilon,L)}$ is a chain homotopy from Δ_L to Δ_ε.

Now apply this to QME_ε.

Recall there are two equivalent forms for the QME_ε:

$$Q I + \frac{1}{2}\{I, I\}_\varepsilon + \hbar \Delta_\varepsilon I = 0$$

$$\Longleftrightarrow$$

$$(Q + \hbar\Delta_\varepsilon) e^{I/\hbar} = 0$$

Want to show the RG flow

$$W(P(\varepsilon, L), I)$$

Satisfies QME_L:

$$0 \overset{?}{=} (Q + \hbar \Delta_L) \exp\left(\frac{1}{\hbar} W(P(\varepsilon, L), I)\right)$$

$$W(P(\varepsilon, L), I) = \hbar \log\left(e^{\hbar \partial_{P(\varepsilon, L)}} e^{I/\hbar}\right)$$

So, WTS

$$\left(Q + \hbar \Delta_L\right)\left(e^{\hbar \partial_{P(\varepsilon, L)}} e^{I/\hbar}\right) \overset{?}{=} 0$$

Use the chain homotopy to commute

$Q + \hbar \Delta_L$ past $e^{\hbar \partial_{P(\varepsilon, L)}}$:

Claim is:

$$\left(Q + \hbar \Delta_L\right) e^{\hbar \partial_{P(\varepsilon, L)}} = e^{\hbar \partial_{P(\varepsilon, L)}} \left(Q + \hbar \Delta_\varepsilon\right)$$

Since $(Q + \hbar \Delta_\varepsilon) e^{I/\hbar} = 0$, this

implies QME_L.

Converse holds because RG flow

is invertible. $\square$

<u>Recall</u> A <u>classical BV thy</u>

is a $-1-$ shifted symplectic vector bundle $E \to M$, & a degree 0

action func

$$S \in \mathcal{O}_{loc}(\mathcal{E})$$

satisfying

$$\{S, S\} = 0$$

S is at least quadratic,

$$S = \underbrace{\langle e, Qe \rangle}_{\text{free}} + \underbrace{I(e)}_{\substack{\text{at least}\\ \text{cubic}}}$$

When $I = 0$, we say this is a free classical BV thy.

It's determined by the triple

$$(\mathcal{E}, \langle -,- \rangle, Q)$$

We assume Q admits a gauge fixing op Q^{GF}.

Def A <u>Perturbative BV QFT</u>

w/ free part $(\mathcal{E}, \langle -, - \rangle, Q)$

& gauge fixing operator Q^{GF} is

a family of effective

interactions $\{I[L]\}_L$, $I[L] \in \mathcal{O}^+(\mathcal{E})[[\hbar]]$

($+$ = at least cubic modulo $\hbar$.) s.t.

- Satisfy RG flow

$$I[L] = W(P(\varepsilon, L), I[\varepsilon])$$

- For each scale L, they satisfy QME_L (i.e. hence for all scales!).

- $I[L]$ has a small L asymptotic expansion in local action funcs.

...

Obstructions to quantization arise from the failure of QME to have a solution at each order in $\hbar$.

Next time (in person!) we will discuss this.